Example 11: Consider the vectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \qquad \boldsymbol{v}_3 = \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_4 = \begin{bmatrix} -4 \\ 2 \\ -5 \end{bmatrix} \tag{9}$$

1. Show that $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) = \text{span}(\mathbf{v}_1, \mathbf{v}_2)$.
2. Find a vector $\mathbf{b}$ in $\mathbb{R}^3$ that is not in $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$.
3. Give a geometric description of the set $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$.

1.) $A = \begin{bmatrix} 1 & 1 & 2 & -4 \\ -1 & 1 & -4 & 2 \\ 1 & 2 & 1 & -5 \end{bmatrix} \overset{\text{exercise}}{\sim} \begin{bmatrix} 1 & 0 & 3 & -3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = C$

"elementary row operations preserve linear combinations"

$$\begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (-1)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0\begin{bmatrix} -3 \\ -1 \\ 0 \end{bmatrix}$$

$$\checkmark \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix} \overset{\checkmark}{=} 3\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + (-1)\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + 0\begin{bmatrix} -4 \\ 2 \\ -5 \end{bmatrix}$$

$\vec{v}_3 = 3\vec{v}_1 - \vec{v}_2 + 0\vec{v}_4$. Hence by theorem 3,

$\text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4) = \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_4)$ $\boxed{A}$

$$\begin{bmatrix} -3 \\ -1 \\ 0 \end{bmatrix} = (-3)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (-1)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\checkmark \begin{bmatrix} -4 \\ 2 \\ -5 \end{bmatrix} \overset{\checkmark}{=} (-3)\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + (-1)\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$\vec{V}_4 = -3\vec{v}_1 - \vec{v}_2$. Hence by theorem 3,

$\text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_4) = \text{span}(\vec{v}_1, \vec{v}_2)$ $\boxed{B}$

$\text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4) = \text{span}(\vec{v}_1, \vec{v}_2)$

Extra Space:

2.) $\left[\begin{array}{cc|c} 1 & 1 & b_1 \\ -1 & 1 & b_2 \\ 1 & 2 & b_3 \end{array}\right] \overset{\text{exercise}}{\sim} \left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & 1 & b_3 - b_1 \\ 0 & 0 & -2b_3 + 3b_1 + b_2 \end{array}\right]$

$0 = -2b_3 + 3b_1 + b_2$

For example, $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $0 = 2$ ↯

$\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is not in the $\text{span}(\vec{v}_1, \vec{v}_2) = \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$

3.) $\text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4) = \text{span}(\vec{v}_1, \vec{v}_2)$ is a plane in $\mathbb{R}^3$.